Approximate Distribution of Nearly Circular Orbits

R E BECKWITH*
University of Southern California, Los Angeles, Calif

The theory of errors is employed to assess the precision of a guidance system designed to place a payload in a prescribed circular orbit Subject to certain hypotheses about the parameters of the vehicle's propulsion and guidance systems at burnout, the form of the distribution of the square of the orbital eccentricity is obtained, and several approximations to it are presented It is indicated how knowledge of this distribution may be used to estimate the probability of mission success, suitably defined, and how this quantity may be used to study various questions that are of interest to the system planner

1 Introduction

FOR certain missions it is required to place a payload in circular or nearly circular orbit at a prescribed altitude No orbit will be perfect, and it is desirable to have available a satisfactory method for predicting the eccentricity of the actual orbit from knowledge of the statistical characteristics of the numerous fundamental sources of error, such as motor characteristics, specific impulse, propellant weight, case weight, guidance errors, etc Such a methodology would be of value in determining to what extent existing equipment and techniques, for which performance data are either available or obtainable through controlled experimentation, are capable of carrying out intricate space missions, and to assess the performance capabilities of proposed systems

The characteristics of the payload's free orbit depend upon the values, at the time of burnout or fuel shutoff, characterizing the preceding group of error sources We will call this set of variables the "fundamental guidance parameters" The guidance and propulsion systems of the vehicle are designed to achieve burnout with the fundamental guidance parameters at certain nominal values so chosen that a free circular orbit at a prescribed altitude is attained

The fact that any orbit will fail to be circular, to a greater or lesser extent, is due to the failure of the fundamental guidance parameters to be exactly at their nominal, or planned, values at the instant of burnout The procedure given in this paper is based on the statistical theory of errors It provides, subject to certain hypotheses about the values of the vehicle's fundamental guidance parameters at the time of burnout, the form of the distribution of the square of the orbital eccentricity, given in terms of the tolerances associated with the (production or guidance) mechanisms influencing the fundamental guidance parameters The exact distribution is difficult to obtain in explicit form, and several approximations to it are given

Illustrations indicate how knowledge of this distribution may be used to provide a rational basis for estimating the probability of mission success, suitably defined, the number of launchings that must be planned in order to achieve, with high probability, at least one successful mission, and related quantities of interest to the system planner

The basic methodology is not limited to the study of "single-shot" launchings; it can be extended to treat synchronous orbits and other more complicated space missions A reasonable application of the procedure would be to assist in the planning of a mission in which a trial orbit is established by a main stage guidance system, subsequent corrections being commanded following analysis of the trial orbit The eventual orbit characteristics can be estimated, based on 1) a prediction interval for eccentricity resulting from the main stage guidance control, and 2) a prediction interval for eccentricity resulting from corrections to be applied after the trial orbit has been analyzed In applications of this type, discrete approximations may be required in the procedure in order to avoid complicated mathematical considerations A byproduct of such an application could be a basis for estimating the weight of terminal correction equipment required aboard the main stage payload, in order to achieve, probabilistically, the desired, nearly circular, final orbit

2 Mathematical Formulation of the Problem

The square of the eccentricity of a free orbit is given[1] by

$$e^2 \equiv f(\beta,r,v) \sin^2\beta + [(rv^2/gR^2) - 1]^2 \cos^2\beta \qquad (1)$$

where β, r, and v, measured at the instant of burnout or fuel shutoff, are the heading angle, measured outward from the normal to the radius vector connecting the center of mass of the vehicle to the center of mass of the earth (origin), the magnitude of the radius vector, and the magnitude of the velocity, respectively; g is the gravitational constant and R is the earth's radius A circular orbit at altitude r_0 (measured from the earth's center of mass) will result when $\beta = \beta_0 = 0$ and $v = v_0 = gR^2/r_0$ In terms of the departures $\Delta\beta = \beta - 0$, $\Delta r = r - r_0$, $\Delta v = v - v_0$ from the planned conditions $\beta_0 = 0$, r_0, v_0 at the instant of burnout, expression (1) can be expanded in a Taylor's series about $(0,r_0,v_0)$ to yield the approximation

$$f(\beta,r,v) \doteq (\Delta\beta)^2 + [(\Delta r/r_0) + (2\Delta v/v_0)]^2 \qquad (2)$$

in which third-order terms, and higher, are omitted

Let W_i, $i = 1(1)n$, denote the values of the fundamental guidance parameters, and let W_{i0} denote their respective nominal values, selected to yield the circular orbit conditions $\beta_0 = 0$, r_0, and v_0 Let $X_i = W_i - W_{i0}$, $i = 1(1)n$, denote the departures of the fundamental guidance parameters from their nominal values The expressions for $\Delta\beta$, Δr, and Δv in terms of the W_i (and hence in terms of the X_i) will be known, either exactly or empirically, for a particular vehicle system Expanding these expressions in a Taylor's series about the nominal values W_{10}, , W_{n0}, and retaining only first-order terms in the "errors" X_i, there result approximations of the form

$$\begin{aligned} \Delta\beta &\doteq \sum_{i=1}^{n} a_{1i}X_i \\ \frac{\Delta r_0}{r_0} &\doteq \sum_{i=1}^{n} a_{2i}X_i \\ \frac{2\Delta v_0}{v_0} &\doteq \sum_{i=1}^{n} a_3 X_i \end{aligned} \qquad (3)$$

Received July 29, 1963; revision received November 7 1963

* Associate Professor, Graduate School of Business Administration

where the coefficients are given by

$$a_{1i} = \partial(\Delta\beta)/\partial W_i$$
$$a_{2i} = (1/r_0)[\partial(\Delta r)/\partial W_i] \quad (4)$$
$$a_{3i} = (2/v_0)[\partial(\Delta v)/\partial W_i]$$

for $i = 1(1)n$, each partial derivative being evaluated at the nominal value $(W_{10}, \quad , W_{n0})$ of the fundamental guidance parameter group

Expressions (3), when substituted into (2), yield the approximation

$$Y = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} X_i X_j \quad (5)$$

to (1), where the coefficients c_{ij} are given by

$$c_{ij} = a_{1i}a_{1j} + (a_{2i} + a_{3i})(a_{2j} + a_{3j}) = c_{ji} \quad (6)$$

for $i, j = 1(1)n$, in terms of coefficients (4)

We introduce a probabilistic aspect into the situation by making the basic assumption that the W_i are a set of independent random variables, W_i being distributed about its nominal value W_{i0} with variance $\sigma_i^2 > 0$, $i = 1(1)n$

With this understanding it is conventional in American industry to assume that an engineering specification for the parameter W_i, of the form $W_{i0} \pm t_i$ (where t_i is a small positive number), conforms to a 99 73% (or $\pm$ 3σ) tolerance interval, whence

$$\sigma_i = \tfrac{1}{3}t_i \qquad i = 1(1)n \quad (7)$$

A slightly different convention is followed in Great Britain [2]

Under the previous assumption the X_i are a set of independent normally distributed random variables, X_i having mean (or expected value) $E(X_i) = 0$ and variance $V(X_i) = \sigma_i^2$, $i = 1(1)n$ The "distribution of nearly circular orbits" is described in terms of the probability distribution of quadratic form (5)

3 Exact Distribution

Write quadratic form (5) as $Y = x'Cx$, in terms of the column vector $x = \text{col}(X_1, \quad , X_n)$, its transpose $x' = (X_1, \quad , X_n)$, and the symmetric matrix $C = (c_{ij})$ defined by elements (6)

The vector random variable x has, by hypothesis, the multivariate normal distribution[3] with mean vector $\theta = (0, \quad , 0)'$ and nonsingular covariance matrix $\Sigma_x = \text{diag}(\sigma_1^2, \quad , \sigma_n^2)$ By elementary theorems in matrix algebra[4] there exist nonsingular P and orthogonal T such that $P\Sigma_x P' = I$, the identity matrix, and $T(P'^{-1}CP^{-1})T' = \text{diag}(\lambda_1, \quad , \lambda_n)$ Here $P = \text{diag}(\sigma_1^{-1}, \quad , \sigma_n^{-1})$; hence $P'^{-1}CP^{-1} = (c_{ij}\sigma_j)$, whence the numbers λ_i are the (real) eigenvalues of the (symmetric) matrix $(c_{ij}\sigma_i\sigma_j)$

In view of the preceding, the random variable $z = (Z_1, \quad , Z_n)'$ defined by

$$z = TPx \quad (8)$$

is distributed according to the multivariate normal law, with mean vector and covariance matrix given by

$$E(z) = \theta \qquad \text{and} \qquad \Sigma = E(zz') = I \quad (9)$$

respectively By the nonsingularity of P and the orthogonality of T, (8) yields $x = P^{-1}T^{-1}z = P^{-1}T'z$, and a direct calculation shows that

$$Y = x'Cx = \sum_{i=1}^{n} \lambda_i Z_i^2 \quad (10)$$

From (9) it is apparent that the Z_i^2 are identically and independently distributed according to the chi-square law with one degree of freedom Thus, quadratic form (5) is distributed as a linear combination of independent chi-square variates, each with one degree of freedom, a special case of a known result [5] The multipliers λ_i in the linear combination are the eigenvalues of the matrix $(c_{ij}\sigma_i\sigma_j)$ Moreover, since $Y \geqslant 0$ is implied by (1), subject, of course, to approximations (2) and (3), it may be asserted that the λ_i are nonnegative numbers

In view of the preceding observations it follows that the characteristic function[6] of the distribution of Y is given by

$$\varphi(t) \equiv E(e^{itY}) = \prod_{j=1}^{n} (1 - 2i\lambda_j t)^{-1/2} \quad (11)$$

The probability density function of Y, $g(y)$ say, can (in principle) be obtained from (11) by application of the Fourier inversion theorem [6] Thus, formally,

$$g(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ity}\,\varphi(t)\,dt \quad (12)$$

is the density function of Y, whereas its distribution function is given by the integral

$$G(y) = \text{prob}\{Y \leqslant y\} = \int_{-\infty}^{y} g(t)\,dt \quad (13)$$

4 Approximation I

Distribution (12) is of a type that resists representation in simple, explicit form except in very special cases Elaborate methods are required and available for a guaranteed close approximation to it,[7–12] all of which involve rather extensive numerical computation Available also are approximation methods based on Gram-Charlier's or Edgeworth's series [13] A different class of approximations results from the approximation of moments of (12) The present paper employs this type of approximation, where the objective is to supply an approximation of sufficient accuracy to answer rather broad questions, yet which is of relative computational simplicity

An obvious approximation to distribution (12) is obtained as follows The expected value and variance of Y are given by

$$E(Y) = \sum_{i=1}^{n} \lambda_i E(Z_i^2) = \sum_{i=1}^{n} \lambda_i \quad (14)$$

and

$$V(Y) = \sum_{i=1}^{n} \lambda_i^2 V(Z_i^2) = \sum_{i=1}^{n} \lambda_i^2 \quad (15)$$

respectively In view of the asserted nonnegativity of the λ_i, approximate distribution (12) by a distribution of the flexible and unimodal gamma family This distributional form is at least suggested by the reproductive property of an important subclass of the gamma family, the chi-square family of distributions

A random variate U is said to be distributed according to the gamma law if its probability density function is of the form

$$h(u) = [1/\Gamma(\alpha + 1)\beta^{\alpha+1}]u^{\alpha}e^{-u/\beta} \quad (16)$$

for $u > 0$, and zero otherwise, where $\alpha > -1$, $\beta > 0$, and $\Gamma(p)$ is the well known gamma function, defined for all real $p > 0$ by the integral

$$\Gamma(p) = \int_0^{\infty} t^{p-1}e^{-t}dt$$

The mean and variance of distribution (16) (no other moments are required to describe this two-parameter family) are given by

$$E(U) = \beta(\alpha + 1) \quad (17)$$

and

$$V(U) = \beta^2(\alpha + 1) \tag{18}$$

respectively The approximation of distribution (12) by distribution (16) is accomplished by equating moments (14) and (15) with moments (17) and (18), respectively The resulting estimates,

$$\bar{\alpha} = \left[\left(\sum_{i=1}^{n} \lambda_i\right)^2 \Big/ 2\sum_{i=1}^{n} \lambda_i^2\right] - 1$$

and

$$\bar{\beta} = 2\sum_{i=1}^{n} \lambda_i^2 \Big/ \sum_{i=1}^{n} \lambda_i$$

are applied to (16), and distribution (12) is approximated by

$$h(y) = [1/\Gamma(\bar{\alpha} + 1)\bar{\beta}^{\bar{\alpha}+1}]y^{\bar{\alpha}}\, e^{-y/\bar{\beta}} \tag{19}$$

for $y > 0$, and zero otherwise

The distribution function of density (19),

$$H(y) = \text{prob}\{Y \leqslant y\} = \int_{-\infty}^{y} h(t)\, dt \tag{20}$$

has been extensively tabled as the "incomplete gamma function "[14]

A further approximation to (12), which is inferior to (19), but supported by the central limit theorem[6] in view of the number n ($n = 20$ to 30) of fundamental guidance parameters encountered in practice, is obtained by assuming that Y is approximately normally distributed with mean and variance given by (14) and (15), respectively Thus, (13) is approximated by

$$\text{prob}\{Y \leqslant y\} = \Phi[(y - \Sigma\lambda_i)/(2\Sigma\lambda_i^2)^{1/2}] \tag{21}$$

where

$$\Phi(u) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{u} e^{-t^2/2} dt \tag{22}$$

is the widely tabled standard normal probability integral Use of approximation (21) should be avoided if

$$\Phi[-\Sigma\lambda_i/(2\Sigma\lambda_i^2)^{1/2}] = 1 - \Phi[\Sigma\lambda_i/(2\Sigma\lambda_i^2)^{1/2}]$$

is not close to zero

5 Approximation II

An approximation to distribution function (13) which is less satisfying than either (20) or (21) but which is more readily computed [requiring only the numbers σ_i obtained from the parameter tolerances through (7) and not the λ_i] is obtained by applying the central limit theorem directly to quadratic form (5)

The random variable Y, defined in (5), has mean and variance symbolized by $\mu \equiv E(Y)$ and $\sigma^2 \equiv E(Y - \mu)^2$, respectively Since the random variables X_i are (assumed to be) normally and independently distributed about zero, X_i having variance $\sigma_i^2 > 0$, $1 = 1(1)n$, the evaluation of μ,

$$\mu = \sum_{i=1}^{n} c_{ii}\sigma_i^2 \geqslant 0 \tag{23}$$

follows directly from the facts that $E(X_i) = 0$ and $E(X_i^2) = \sigma_i^2$ The nonnegativity follows upon expressing the c_{ii} in terms of the coefficients a_{ki} in (6) The evaluation of σ^2,

$$\sigma^2 = 2\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^2\sigma_i^2\sigma_j^2 \tag{24}$$

follows (after some reduction) from the same facts, together with $E(X_i^3) = 0$ and $E(X_i^4) = 3\sigma_i^4$

By the central limit theorem, the distribution of Y is approximately normal, with mean and variance given by (23) and (24), respectively That is to say,

$$\text{prob}\{Y \leqslant y\} = \Phi[(y - \mu)/\sigma] \tag{25}$$

in terms of function (22) As with approximation (21), use of (25) should be viewed with suspicion if $\Phi(-\mu/\sigma) = 1 - \Phi(\mu/\sigma)$ is not close to zero; i e , unless $\mu \geqslant 3\sigma$, approximately

A bound can be placed on μ/σ, hence on $\Phi(-\mu/\sigma)$, by means of which the general utility of approximation (25) can be assessed Express μ^2 and σ^2 in terms of the coefficients a_{ij} of (6),

$$\mu^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} [a_{1i}^2 + (a_{2i} + a_{3i})^2] \times [a_{1j}^2 + (a_{2j} + a_{3j})^2]\sigma_i^2\sigma_j^2$$

$$\sigma^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} [a_{1i}a_{1j} + (a_{2i} + a_{3i})(a_{2j} + a_{3j})]^2\sigma_i^2\sigma_j^2$$

and observe that, by Schwarz's inequality (for sums rather than integrals), $2\mu^2 \geqslant \sigma^2$; hence by (23), $\mu/\sigma \geqslant \frac{1}{2}2^{1/2}$†; thus, $\Phi(-\mu/\sigma) \leqslant 0\ 24$ (approximately), at the very worst

Data from actual systems should, in any particular case, be used to indicate whether or not approximation (25) may safely be applied to distribution function (13) in place of the more exact (and more difficult to compute) forms (20) and (21)

6 The Case of Nonindependence

In the preceding treatment it has been assumed that "errors" X_i are mutually independent Although this may approximately be the case in many vehicle systems, it is not necessarily so For example, if W_{10} is the nominal voltage and W_1 the actual voltage provided by a power supply, and W_{20} and W_2 are, respectively, the nominal and actual burnout values of a critical parameter controlled by a device driven by the power supply, then a positive "error" in voltage, $W_1 - W_{10} = X_1 > 0$, may necessarily be accompanied by a positive (or negative) "error" $W_2 - W_{20} = X_2$, the opposite holding for a negative X_1

For full generality of the treatment, provision should be made for possible nonindependence among the random variables X_i, $i = 1(1)n$ Since the treatment given here has been limited to orbit parameter errors that are linear in the X_i, in (3) we restrict our characterization of nonindependence to pairwise (linear) correlation among the X_i

The covariance matrix $\Sigma_x = E(xx')$ introduced in Sec 3 has general element $\sigma_{ij} = E(X_iX_j)$, with σ_{ij} not necessarily zero for $i \neq j$ (as was hypothesized in Sec 3) and $\sigma_{ii} = \sigma_i^2$ Observe that Σ_x may no longer, in general, be considered a diagonal matrix

Subject to the plausible restriction that the joint distribution of the X_i be nondegenerate (rendered even more plausible by the fact that the restriction follows, in the case of independence, from $\sigma_i^2 > 0$), it follows, as before, that quadratic form (5) is distributed as a linear combination of independent chi-square variates, each with one degree of freedom

The only practical difference between the situation when the X_i are mutually independent, and when linear correlations exist, is that the matrix P of Sec 3 no longer has the simple form diag (σ_1^{-1}, , σ_n^{-1}), and must be computed by standard methods The numbers λ_i in (10) are, of course, the eigenvalues of the matrix $P'^{-1}CP^{-1}$ All equations and expressions remain unchanged

† This inequality can be obtained by other means for $n \geq 1$ and $\lambda_i = 1$ from the exact distribution of Y

Estimation of the elements of covariance matrix Σ_x may best be discussed within the context of a particular application, although certain general comments can be made

The diagonal elements σ_{ii} are the squares of the quantities σ_i, related by (7) to the tolerances of the fundamental guidance parameters Many of the off-diagonal elements σ_{ij} can be assumed to be zero where no logical or experimentally observed connection between the fundamental guidance parameters W_i and W_j exists For certain pairs of fundamental guidance parameters, theoretical or empirical relationships connecting the "errors" may be known which can be linearized (within the anticipated ranges of values) to provide estimates of the corresponding covariance terms

For other pairs of fundamental guidance parameters it will be necessary to conduct engineering experiments in order to determine the extent to which the associated "errors" are linearly related Here, the covariances can be estimated by standard methods of (multivariate) linear regression analysis [3 5 6 15] Such experimentation may reveal that the associated X_i are not unbiassed [i e , that some $E(X_i) \neq 0$], thus necessitating revision of the corresponding nominal values, or that certain of the σ_i are not consistent with hypothesized relationship (7), thus overriding it

7 Applications

1) Suppose that an orbital mission will be considered "successful" if the eccentricity of the orbit is less than a prescribed small positive number δ in absolute value (i e , if the orbit is very nearly circular) For a given set of vehicle characteristics, what is the probability that a single shot will result in a successful mission? The probability of success on a single shot, i e , the probability that $-\delta < e(W_1, \ldots, W_n) < \delta$, is given by $\text{prob}\{0 \leqslant Y < \delta^2\}$, which may be approximated, variously, by (20), (21), or (25)

2) Suppose, in application 1, that it is necessary to estimate the number m of launchings (assumed to be statistically independent) required in order that the probability of program success (defined as at least one successful mission) be at least $1 - \alpha$, where α is a small positive number Denote $\text{prob}\{0 \leqslant Y < \delta^2\}$ by p Then m is the least integer greater than or equal to the ratio $\ln\alpha/\ln(1 - p)$ In view of the "learning process" associated with missile launchings (a type of nonindependence) it would be proper practically, if not mathematically, to view the integer m so obtained as providing somewhat more protection than that stated in the formulation of the problem

3) A $100(1 - \alpha)\%$ prediction interval for the square of the orbital eccentricity resulting from a single launching when a circular orbit is planned is given by $y_{1-\alpha}$, a number determined by the equation

$$G(y_{1-\alpha}) = 1 - \alpha \tag{26}$$

obtained from (13), which may be approximated by (20), (21), or (25), as appropriate Care should be observed in attempting to interpret (26) in terms of the orbital eccentricity, the square root of Y

4) Suppose that a limited budget is available for improving the mission capability of an existing rocket system Where should the funds be invested in order to provide the greatest over-all benefit? Let us simplify the problem by ignoring all emotional, nonquantitative, and state-of-the-art considerations, and perform a sensitivity analysis designed to identify the fundamental guidance parameter tolerances that most strongly influence mission capability The motivation for doing this is that, other considerations aside, technological improvement in the mechanisms characterized by these tolerances will yield the greatest immediate improvement in system capability One approach to treating this problem in terms of the situation posed in application 1, e g , is to expand the expression for $\text{prob}\{0 \leqslant Y < \delta^2\}$ [which, with approximation (21) or (25), is related to the well-known "error function"] in a convergent infinite series Termwise differentiation of this series, characterized by

$$\text{prob}\{0 \leqslant Y < \delta^2\} = \Omega(s_1, \ldots, s_n) \tag{27}$$

with respect to the variables $s_i = \sigma_i$ (or, equivalently, s_i any monotone function of σ_i, such as σ_i^2 or $\ln\sigma_i$), and evaluation at the nominal values $W_{10}, \ldots, W_{n0}$ of the fundamental guidance parameters, yields the sequence of numbers

$$\partial\Omega/\partial s_i|_{(W_{10} \ldots W_{n0})} \qquad i = 1(1)n \tag{28}$$

The indices i providing the largest absolute values of numbers (28) identify the error sources which may be regarded as the most promising candidates for technological improvement [it being supposed that technological "improvement" will reduce the tolerances t_i, and through (7) the associated σ_i] in the interest of efficient over-all system improvement

References

[1] Thomson, W T , *Introduction to Space Dynamics* (John Wiley and Sons, Inc , New York, 1961), Chap 4

[2] Beckwith, R E , Tolerance limits for the output of systems of cascaded elements," Inst Radio Engrs Trans Reliability Quality Control **RQC-11**, 54–58 (July 1962)

[3] Anderson, T W , *An Introduction to Multivariate Statistical Analysis* (John Wiley and Sons, Inc , New York, 1958), Chaps 2 and 3

[4] Bellman, R , *Introduction to Matrix Analysis* (McGraw Hill Book Co , Inc , New York, 1960), Chap 4

[5] Scheffé, H , *The Analysis of Variance* (John Wiley and Sons, Inc , New York, 1959), Appendix V

[6] Cramér, H , *Mathematical Methods of Statistics* (Princeton University Press, Princeton N J , 1946), Chaps 10 and 17

[7] Robbins, H E , "The distribution of a definite quadratic form," Ann Math Stat **19**, 266–270 (1948)

[8] Gurland, J , Distribution of definite and indefinite quadratic forms," Ann Math Stat **26**, 122–127 (1955)

[9] Pachares, J , "Note on the distribution of a definite quadratic form, Ann Math Stat **26**, 128–132 (1955)

[10] Grad, A and Solomon, H , "Distribution of quadratic forms and some applications,' Ann Math Stat **26**, 464–477 (1955)

[11] Solomon, H , On the distribution of quadratic forms in normal variates," *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability* (University of California Press Berkeley and Los Angeles, 1961), Vol I, pp 645–653

[12] Jones, R H , ' Spectral estimates and their distributions " Skand Aktuar **45**, Part I, 39–69, Part II, 135 153 (1962)

[13] Kendall, M G and Stuart A , *The Advanced Theory of Statistics* (Hafner Publishing Co , New York, 1958), Vol I, Chap 6

[14] Pearson, K , *Tables of the Incomplete Gamma Function* (Cambridge University Press, London, 1922)

[15] Kendall, M G and Stuart, A , *The Advanced Theory of Statistics* (Hafner Publishing Co , New York, 1961) Vol II Chap 26